Table 1 Solutions of the displacements at the unsupported corner of a square-shaped plane strain panel (10^{-3} in.)

Element	Hybrid stress model		Key's principle		Displacement model bi-linear		Displacement model four subtriangles	
ν \ mesh	1×1	2×2 4×4	1×1	2×2	1×1	2×2	1×1	2×2
0.33	0.191	0.185	0.201	0.186	0.178	0.182	0.170	0.183
0.40	0.178	0.172	0.194	0.173	0.150	0.164	0.161	0.167
0.45	0.164	0.160	0.188	0.160	0.110	0.139	0.114	0.152
0.48	0.153	0.151	0.183	0.150	0.634×10^{-1}	0.106	0.130	0.142
0.4999	0.145	0.144	0.180	0.143	0.479×10^{-3}	0.189×10^{-2}	0.120	0.133
0.49995	0.145	0.144	0.180	0.143	0.240×10^{-3}	0.953×10^{-3}	0.120	0.133

These are again the incompressibility constraints. It is seen that the number of constraints is equal to the number of stress parameters β. To restrict the number of constraints to a minimum it is desirable to assume constant mean stress σ_m for the entire element. In that case the constraint condition is again the maintaining of constant volume of the element. However, if the mean stress distribution is assumed linear in x and y there will exist three kinematic constraints which are exactly the same as Eq. (2). The assumption of constant mean stress for the entire element is, however, unreasonable if the deviatoric strains contain higher order terms.

Example Problem

A plane strain problem was solved and double precision was used to minimize the round off error. The structure is 1″ × 1″ square planform with one horizontal edge clamped and the neighboring edge free to slide. The other horizontal edge is acted on by a uniform tensile stress of 4000 psi. Different rectangular elements used are: 1) four mode assumed stress hybrid element with seven stress parameters; 2) four node element with bilinear displacement assumption; 3) an element made of four constant strain triangles forming two diagonals, and 4) four node element derived by Key's principle with constant mean stress.

It should be noted that the scheme of subelement construction for the third element above is not applicable to other types of elements such as axisymmetric solids and three-dimensional solids. Solutions for the vertical displacement at the unsupported corner are given in Table 1. Here the assumed stress hybrid model using 2×2 and 4×4 meshes are identical hence they can be considered as the reference solutions of the problem.

It is seen that the assumed stress hybrid model yields the most accurate solutions for all values of ν, although the element with four subtriangles and the one by Key's principle also led to reasonable solutions for ν approaching 0.5. On the other hand, in the case of element based on bilinear displacement, the boundary conditions of the present problem will prevent any deformation on account of the severe constraints due to the incompressibility condition.

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Technical Comments

Comment on "Extended Integral Equation Method for Transonic Flows"

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Introduction

NIXON[1] has reported on a method of analysis for calculating transonic flows which he describes as the extended integral equation method. This method, he states, results in a considerable improvement in the accuracy with respect to the "standard" integral equation which, on the other hand, yields unsatisfactory results. To visualize the gain in accuracy Nixon compares one of his obtained solutions with a "standard" integral equation solution[2,3]. The purpose of the present Note is threefold: to show that the comparison presented by Nixon is erroneous and, hence, misleading; to bring some results which qualify the validity of the approximation made in the evaluation of the subject field integral; and to comment on an earlier criticism[4] of related nature.

Analysis

The problem under discussion concerns the solutions of the nonlinear potential equation which describes the transonic small-disturbance flow past a slender profile. The integral equation method formulates the solution to this partial differential equation as an integral equation in a reduced space of the physical coordinates. Similarity solutions are then obtained, but the analysis requires a knowledge of the perturbation velocities in the flowfield. These can, however, be related to the velocities at the profile surface in an approximated manner of various degrees of accuracy.

One example chosen by Nixon to point out the feature of the extended integral equation method is the supercritical flow

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($M_\infty = 0.87$) past a 6% thick parabolic arc airfoil at zero angle of attack. To compare his results with other available solutions Nixon utilizes the fully conservative ($\Delta x = 0.001$) similarity solution given by Murman[5] for a similarity parameter $K = (1 - M_\infty^2)/(M_\infty \ \tau^{2/3})$ equal to 1.8. Thus, for a thickness ratio of $\tau = 0.06$ the corresponding freestream Mach number is $M_\infty = 0.8715$, i.e., slightly higher than the value used by Nixon in his Fig. 3. In converting Murman's pressure data, given by the reduced pressure coefficient $-\bar{C}_p = -\bar{C}_p\ (x)$, it is consistent with his analysis to use the relationship $c_p = \tau^{2/3} M_\infty^{\ -3/4} \bar{C}_p$ which for $\tau = 0.06$ and $M_\infty = 0.8715$ will yield $c_p = 0.1699 \cdot \bar{C}_p$.

Nixon in his Fig. 3 also compares his solution ($\tau = 0.06$; $M_\infty = 0.87$) with the results of Ref. 2 which are based on a thickness ratio of $\tau = 0.084$ and a Mach number at infinity of $M_\infty = 0.8483$. The similarity parameter used in this work is expressed as a reduced thickness parameter $\tau_R = (\kappa + 1) M_\infty^2 (1 - M_\infty^2)^{-3/2} \tau$ (see Eq. (5b) of Ref. 2), and this parameter will attain the value $\tau_R = 0.9771$ for the stated values of τ, M_∞ and $\kappa = 1.4$. Hence, the solution of Ref. 2 can also be looked upon as the solution to the flow past a 6% thick airfoil at $M_\infty = 0.8754$, i.e., both profiles satisfy the same boundary conditions. Furthermore, the given pressure data (c_p – values) must be transformed to this different flow condition by the use of the reduced pressure coefficient $c_{p_R} = [(\kappa + 1) M_\infty^2]^{1/3} \tau^{-2/3} c_p$ (see p. 61 of Ref. 2). Since under such a comparison the reduced pressure coefficient must remain constant one obtains the relation

$$(c_p)_{\tau = 0.06; M_\infty = 0.8754} = 0.7825 \cdot (c_p)_{\tau = 0.084; M_\infty = 0.8483} \qquad (1)$$

The use of the same reduction formulas as applied to the data of Murman, i.e., $c_p = \tau^{2/3} M_\infty^{\ -3/4} c_{pR}$, would produce a change of the factor appearing in Eq. (1) to the value 0.7804. Which reduction is used in Ref. 1 is, however, unclear since that work contains an undefined transonic parameter k.

Figure 1 shows a similar comparison of results as given by Nixon[1] in his Fig. 3. The solutions of Refs. 5 and 2, however, have been reduced according to the relations previously discussed. Although the deviation in pressure levels is less pronounced, the various locations of the shock are the same as in Nixon's figure. This is also correct since a similarity transformation does not alter the shock locations. However, a small increase in the freestream Mach number (as it is indicated in Fig.1) can move the shock downstream by the same order of magnitude as the difference depicted in the figure. A direct comparison (i.e., same thickness ratio and same Mach number at infinity) would therefore be mandatory for an evaluation of the relative accuracy of the integral equation solutions, and it is surprising that this has not been shown in the first place.

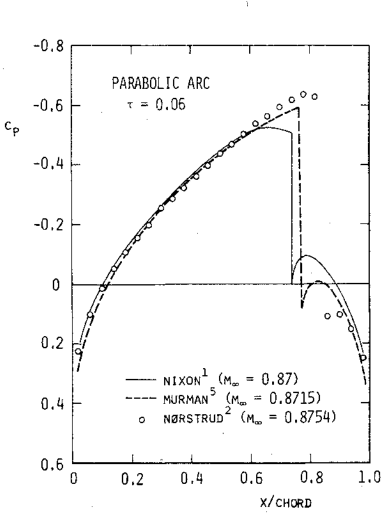

Fig. 1 Pressure distribution on a parabolic arc airfoil in supercritical flow with shocks.

Another example given by Nixon is the subcritical flow past the same parabolic arc airfoil. The comparative solution chosen in this example is taken from Murman and Cole[6] which states the flow conditions through the similarity parameter $K_s = (1 - M_\infty^2)\ (M_\infty^2 \tau)^{-2/3}$. For a 6% thick airfoil a value of $K_s = 3.0$[6] would imply $M_\infty = 0.8085$. Nixon on the other hand operates with a Mach number of $M_\infty = 0.8$ ($K_s = 3.1628$). If now the reduction $c_p = (\tau / M_\infty)^{2/3} \bar{C}_p$ of Ref. 6 is applied to transform the pressure data to the correct combination of $\tau = 0.06$ and $M_\infty = 0.8085$ one would obtain the relation $c_p = 0.1766\ \bar{C}_p$. The results are shown in Fig. 2 and a small modification of Nixon's Fig. 2 can be observed. It is realized that some inaccuracies will occur when transforming data on a graphical basis, but the point to be made here is that a matching of Mach numbers would bring the two solutions further apart.

To bring in the topic discussed by Nixon which concerns the evaluation of the field integral, a third (approximate) solution is added to the case of Fig. 2. Nørstrud[7] has given a correction factor for compressible flow which depends on the incompressible solution, $u_i(x)$, and on the local curvature of the profile surface. With the incompressible solution for a parabolic arc profile given by

$$\frac{u_i(x) - u_\infty}{u_\infty} = \frac{2\tau}{\pi} \left[2 - (1 - 2x) \ln \frac{1 - x}{x} \right] \qquad (0 < x < 1) \qquad (1)$$

the calculated pressure distribution $c_p = c_p(x)$ for $M_\infty = 0.8085$ is plotted in Fig. 2. Even though the method is based on a linear decay of the velocity normal to the freestream direction and on the fact that the compressible solution must be identical (i.e., $= u_\infty$) to the incompressible solution $u_i(x)$

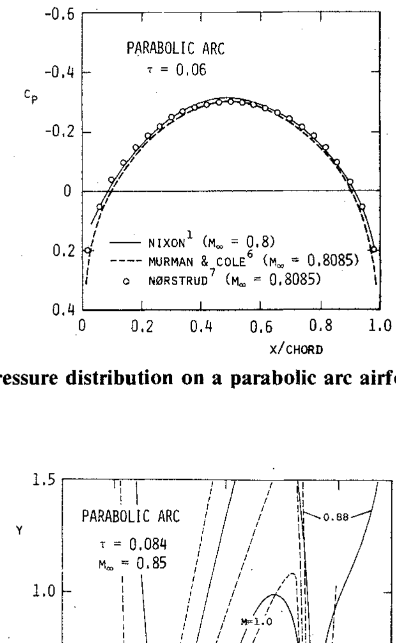

Fig. 2 Pressure distribution on a parabolic arc airfoil in subcritical flow.

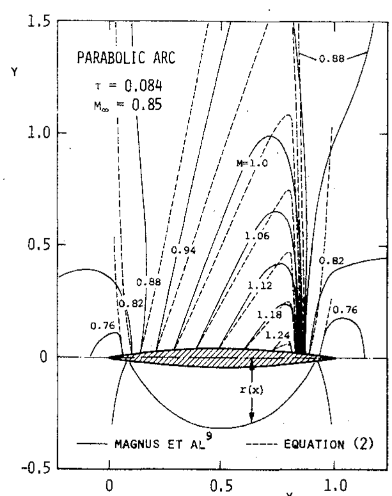

Fig. 3 Mach line contours around a parabolic arc airfoil in supercritical flow with shocks.

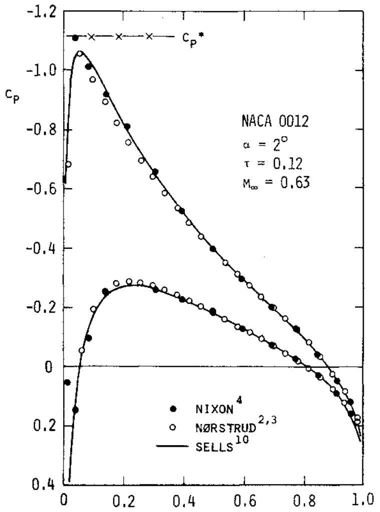

Fig. 4 Pressure distribution on a lifting NACA 0012 airfoil in subcritical flow.

at the points $x = 0.0832$ and 0.9168, the agreement away from these singular points is satisfactory. It should be noted that the maximum value of the pressure coefficient (at $x = 0.5$) is -0.2980 for $M_\infty = 0.8085$ (and $\tau = 0.06$), whereas $c_p = -0.2902$ for $M_\infty = 0.8$.

The solutions of Nφrstrud as presented in Figs. 1 and 2 are based on some approximation in the description of the flowfield (see Ref. 8 for a more detailed discussion). In particular, the solution[2] of Fig. 1 is obtained by describing the flowfield according to the functional relationship for the velocity decay

$$u(x, [1-M_\infty^2]^{1/2} y) = u(x,0) \exp\left[-(1-M_\infty^2)^{1/2} y/r(x)\right] \quad (2)$$

where the function $r = r(x)$, which depicts the lateral extent of the compressibility effects, is a universal function for a given airfoil shape. For a parabolic arc profile one can write $r(x) = |[u_i(x) - u_\infty)/(4\tau u_\infty)]|$ and with Eq. (2) one can evaluate $u(x,y)$ as a function of $u(x,0)$. This has been done in connection with the data given by Magnus et al.[9] and the results are shown as Mach contours in Fig. 3. For values of $|y| < r$ the comparison of field values is good, at least, it shows a qualitative agreement on which some conclusions can be based. Since Nixon in his method[1] uses field points as solution points, it would be of fundamental interest to see a direct comparison of calculated and assumed decay functions.

The last comment to be made is only indirectly related to the work presented in Ref. 1. The present author does agree with Nixon when he states that the approximate evaluation of the field integral in the "standard" integral equation method is adequate for subcritical flows.[1] This is, e.g., shown in Fig. 4 where a comparison is made between the extended[4] and a standard[2,3] integral equation solution together with the "exact" solution[10] obtained by Sells method. However, a conflict occurs when Nixon argues that the method of solution presented in Ref. 3 for lifting flows is incorrect and nonunique[4]. To present a formal disputal is difficult since Nixon[4] starts his discussion with an equation which is not found in Ref. 3. This is further complicated by sign errors in an equation (labeled 5 in Ref. 4) which is derived from this starting equation.

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Reply to Author to H. Nφrstrud

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IN his Comment, Nφrstrud makes three points, namely: 1) that the comparison of the method presented in Ref. 1 with other methods[2-4] is erroneous and misleading; 2) that the statement made on the validity of the approximations used in the 'standard' integral equations method may be misleading; 3) that the argument present in Ref. 5 regarding the formulation of the integral equations for lifting flows is incorrect. These points will be considered in sequence.

The basic differential equation used in Ref. 1 is

$$\bar\phi_{\bar x\bar x} + \bar\phi_{\bar z\bar z} = \bar\phi_{\bar x}\bar\phi_{\bar x\bar x} \quad (1)$$

where, if M_∞ is the freestream Mach number, $k\ (M_\infty)$ is a transonic parameter, $\beta = (1-M_\infty^2)^{1/2}$, and γ is the ratio of specific heats,

$$\bar\phi = \frac{k(\gamma+1)}{\beta^2}\phi, \ \bar x = x, \ \bar z = \beta z \quad (2)$$

where $\phi\ (x,z)$ is the perturbation velocity potential and (x,z) is the Cartesian co-ordinate system in real space.

The tangency boundary condition for Eq. (1) is

$$\bar\phi_z(\bar x, +0) = [(\gamma+1)k(M_\infty)/\beta^3]\tau z_T'(x) \quad (3)$$

where τ is the thickness/chord ratio of the aerofoil, and $z_T(x)$ is a shape function for a family of aerofoils.

A solution for the velocity variable $\bar u(\bar x,\bar z)\ (=\bar\phi_{\bar x}(\bar x,\bar z))$ of the problem defined by Eqs. (1,3) is dependent only on the shape function $z_T(x)$ and the parameter

$$(\gamma+1)k(M_\infty)\tau/\beta^3$$

If the shape function is the same in two cases and if the parameter $[(\gamma+1)k(M)_\infty)\tau/\beta^3]$ has the same value in both cases then the scaled velocity $\bar u(\bar x,\bar z)$ is the same. Since $(\gamma+1)$ is a constant $[(\gamma+1)=2.4]$, the similarity condition for $\bar u(\bar x,\bar z)$ is that

$$k(M_\infty)\tau/\beta^3 = \text{constant} \quad (4)$$

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